J. Fluid Mech. (1966), *vol.* 26, *part* 2, *pp.* 411-431
Printed in Great Britain

Dynamics of rotating fluids: a survey

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(Received 25 April 1966)

The dynamics of rotating fluids was, in the main, developed by methods special to the field, using the equations of motion of a fluid in a rotating frame of reference. It is, nevertheless, possible to derive all the leading results from the classical principles of fluid dynamics in non-rotating frames; specifically, from the rules governing rate of change of vorticity. Although writers on the subject have adopted this approach increasingly often in recent years, the author believes that a broad survey of the field, deriving results from those classical rules concerning vorticity, has not previously been given and may be of some interest to fluid dynamicists in general.

The present survey was read to the IUTAM Symposium on Rotating Fluid Systems at La Jolla, California, on 28 March 1966. It states briefly (§2) the rules governing rate of change of vorticity, and then applies them, first, to problems of steady relative motion of rotating fluids; in particular, of the atmosphere (§3), of rotating fluids in the laboratory (§4), and of the oceans (§5). Waves and wavy movements are then studied, first (§6) for systems with constant Coriolis parameter, including inertial waves, surface waves, 'long waves' and internal waves, and, secondly (§7), for systems with variable Coriolis parameter, including Rossby–Haurwitz waves with and without the influence of tidal effects, as well as problems of barotropic and baroclinic instability. Vorticity principles are used as the sole theoretical tool throughout the survey.

1. Introduction

I am seized with fear and dismay as, in a room including the world's greatest exponents of the dynamics of rotating fluids, in a room indeed where I am probably the only individual who has never written on the subject, I rise to give an introductory survey of this formidably difficult field. Frequently during the last few months I have asked myself whether the organizers of this conference, when they chose someone whose research work has been exclusively in other areas of fluid mechanics to survey before such an audience such a specialized branch of that science, were motivated by pure desire to inflict terror; or whether they merely wanted to make the great ones, who together have created the dynamics of a rotating fluid, free to concentrate during the next few days on expounding their latest researches.

At most one other possible motive existed, namely, to reduce the specialized appearance of the subject by having it surveyed by someone who could only look at it from the point of view of the general science of fluid dynamics, and might attempt to fit the specialized results on rotating fluids that those in this room have discovered into the general framework of fluid-dynamic knowledge. Whether or not this charitable reading of the organizers' intentions was the correct one, it seemed the only basis on which I could prepare something, and accordingly, I intend to survey different parts of the field in this way, concentrating in the first half of my talk on vorticity considerations in steady flow, and in the second on wave propagation. But please be merciful to the novice in your midst!

The subject appears to be divided into the mechanics of 'thin' sheets of fluid in rotation (this includes the cases of the ocean and the atmosphere, in the sense that their depth is small relative to their horizontal extent) and the mechanics of 'fat' bodies of rotating fluid such as one may use in the laboratory to demonstrate phenomena like inertial waves and Taylor columns. However, there is an important continuity of ideas between the 'fat bodies' and 'thin sheets' cases.

Early work in both fields was founded on the equations of motion of a fluid in a rotating frame of reference, but a lot of recent work has, instead, used results from the classical dynamics of fluids in non-rotating frames, including results on rate of change of absolute vorticity in some form such as

$$\frac{D}{Dt}\left(\frac{\zeta+f}{H}\right) = 0. \tag{1}$$

These approaches through the properties of vorticity have proved very useful and it is of interest therefore to see how much of the dynamics of a rotating fluid can be understood quantitatively from this point of view.

2. Properties of vorticity in non-rotating frames

In a non-rotating frame of reference, vorticity behaves in the following way*. First, when external forces are conservative and compressibility and viscosity can be neglected, the rate of change of vorticity is given by a simple geometrical rule. We imagine the fluid flow at one instant as including a multitude of tiny arrow-shaped elements of fluid, each pointing in the direction of the local vorticity $\boldsymbol{\omega}$ and with length proportional to its magnitude. Then these individual arrow-shaped elements of fluid must move thereafter so that they continue to specify both the direction and (through their lengths) the magnitude of the vorticity at every point.

When density change cannot be neglected but takes place either adiabatically or isothermally, that is with either the specific entropy or the temperature uniform and constant, this result is still valid but with the lengths of the arrows proportional to vorticity divided by density. When, however, conditions are neither adiabatic or isothermal, the rate of change of vorticity differs from that given by the rule by a term, tangential to an isobaric surface, of magnitude proportional to the derivative of specific entropy or of temperature along a perpendicular tangent. In meteorological and oceanographic problems this means to good approximation that only horizontal vorticity is generated, at a rate

$$g\alpha(\partial T/\partial y, -\partial T/\partial x, 0), \tag{2}$$

where α is the coefficient of expansion of the fluid at constant pressure.

* For an account of vorticity and its properties derived from basic mechanical principles, see Lighthill (1964).

When viscosity cannot be neglected, but the flow remains laminar, the basic rule is modified by a diffusion of vorticity at a diffusion coefficient equal to the kinematic viscosity. Turbulent flow produces, to a very rough approximation, a similar diffusion of mean vorticity, but of course at a much larger diffusion coefficient. This 'eddy diffusivity' is not constant, diminishing, for example, near solid boundaries. Diffusion of vorticity is particularly important near solid boundaries since new vorticity, that is continually generated at a boundary when fluid flows over it, diffuses outward to form a boundary layer.

Lastly, when the external force field per unit mass (X, Y, Z) is not conservative, the rate of change of vorticity contains yet another term, equal to its curl.

I have stated rather baldly, then, the basic result that arrow-shaped elements of fluid, each pointing in the direction of the local vorticity $\boldsymbol{\omega}$ and with length proportional to ω (or, more generally, to ω/ρ) move in such a way that they continue to specify the vorticity field, and I have indicated how temperature gradients, diffusion and non-conservative external forces may modify the result. Let me show now how these facts are applicable to the dynamics of rotating fluids.

3. Application to elementary atmospheric dynamics

The rotation of a system containing fluid, with angular velocity $\boldsymbol{\Omega}$, imparts a vortcity $2\boldsymbol{\Omega}$ to each element of fluid, additional to the vorticity $\boldsymbol{\omega}$ of that element's relative motion in the rotating system. All the above rules must therefore be applied to the sum $2\boldsymbol{\Omega}+\boldsymbol{\omega}$, usually called the absolute vorticity.

In meteorology and oceanography the planetary component of vorticity, $2\boldsymbol{\Omega}$, has magnitude $1{\cdot}45 \times 10^{-4}\,\text{sec}^{-1}$. This may actually be quite small in the atmosphere compared with the vorticity of the relative motions, whose *horizontal* component at any rate is typically one order of magnitude greater, rising to several in the atmospheric boundary layer.

However, the vertical component of vorticity of the relative motions, usually written ζ, is normally not nearly so big, being comparable with or smaller than the planetary vorticity for all the larger-scale motions. This fact is important because the rules I stated regarding vorticity changes effect changes in the vertical component almost independently of changes in the horizontal component in many situations.

These include all situations when horizontal lines of fluid perpendicular to the flow remain nearly horizontal. The vertical distance H between two such lines can, however, change, as in the flow over some topographical feature. Figure 1 shows that an arrow representing the absolute vorticity vector will then change its vertical component in direct proportion to H. This vertical component is $\zeta+f$, where

$$f = 2\Omega \sin\theta \tag{3}$$

(with θ the latitude) is the 'Coriolis parameter' or vertical component of planetary vorticity, so that these are circumstances in which (neglecting diffusion and non-conservative forces) its rate of change is governed by equation (1).

In briefly surveying the application of this equation in meteorology, I start with the simplest approximation that is made, namely, the geostrophic approximation with f treated as constant. The geostrophic approximation implies that ζ is negligible compared with f (so that a Rossby number U/Lf is small) and that vertical movements do not occur. The equation then becomes $Df/Dt = 0$, which for constant f is a mere identity.

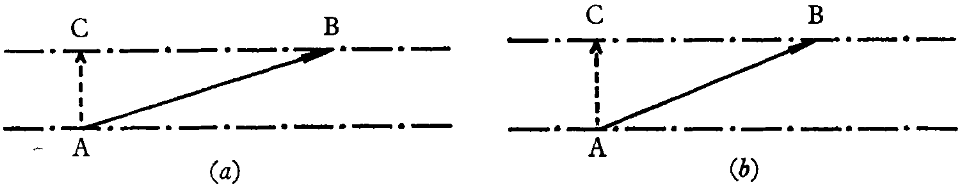

FIGURE 1. Flow viewed from downstream (*a*) at time t, (*b*) at time $t+\delta t$. —·—·—, Pair of nearby horizontal lines of fluid perpendicular to the flow; A——→B arrow-shaped element in direction of vorticity, with length proportional to its magnitude; A– – – →C vertical resultant of element AB, with length proportional to vertical component of vorticity.

Thus there is an inherent degeneracy, as the Rossby number tends to zero, in the equations of two-dimensional motion of a 'thin sheet' of fluid with the Coriolis parameter constant. Such a degeneracy is a familiar feature of the geostrophic wind approximation, since it allows an absolutely *arbitrary* set of lines of constant pressure to act as streamlines for a steady velocity field

$$f^{-1}(-\partial p/\partial y, \partial p/\partial x).$$

We shall see that the degeneracy is important in other contexts also.

In the *finite*-Rossby-number *gradient*-wind approximation, changes in f and H are still neglected, but ζ is not neglected compared with f. The equation (1) then becomes $D\zeta/Dt = 0$, the ordinary equation for two-dimensional motions of an incompressible inviscid fluid. Steady solutions include, for example, circular vortices. When in addition the variation of f with *latitude* is taken into account, then poleward moving air loses cyclonic vorticity and equatorward moving air gains it and the streamlines in anticyclones are bunched together on the poleward side, and in cyclones on the equatorward (Høiland 1950).

In other problems the variation of H may be more important. In tropical cyclones a great vertical stretching occurs, producing large increases in $\zeta+f$, so that the vertical component of vorticity is enormously increased above its planetary value by stretching. Flow over mountain ranges, on the other hand, can locally reduce H and hence $\zeta+f$, so that anticyclonic relative vorticity forms in a sheet over the ridge, permitting the velocity parallel to the ridge to be discontinuous across it.

But I want to go back now and consider changes in *horizontal* velocity, that is, in wind *shear*. The eastward component of vorticity ξ is the rate of change of north wind with height, and the northward component η is the rate of change of west wind with height. These shears produce their own twisting effect on the arrows representing vorticity vector. Indeed, they produce a conversion of vertical vorticity into horizontal vorticity, at a rate

$$[\eta(f+\zeta), \ -\xi(f+\zeta), 0].$$

For example, the northward component of vorticity η is the rate of increase of west wind with height, and this increases the eastward component of vorticity ξ at a rate of $\eta(f+\zeta)$. Any horizontal vorticity helps, then, to create horizontal vorticity in the direction at right angles.

This does not imply that no steady wind with shear is possible. Where there is a horizontal temperature gradient, it can by equation (2) generate equal and opposite vorticity in the perpendicular direction if

$$\eta(f+\zeta) = -g\alpha(\partial T/\partial y). \tag{4}$$

When ζ is neglected compared with f this is the familiar thermal-wind equation relating the increase of west wind with height to southward temperature gradient.

Again, the tendency of horizontal vorticity to create horizontal vorticity at right angles is balanced in the Ekman layer by diffusion of vorticity. On the geostrophic approximation this balances

$$\eta f + \nu(\partial^2\xi/\partial z^2) = 0, \quad -\xi f + \nu(\partial^2\eta/\partial z^2) = 0, \tag{5}$$

where ν is the eddy diffusivity.

These equations determine the fate of the vorticity generated at the ground when the wind blows over it and they make clear both the physical reason for the Ekman spiral, and for the well-known dependence on $\exp[-(1+i)z\sqrt{(f/2\nu)}]$ in the case ν constant.

Both the effects just noted are present in a cold front. Outside the Ekman layers near the ground there are definite discontinuities Δu in tangential velocity and ΔT in temperature at the front. Horizontal vorticity tangential to the front is created at rates ηf by vortex-line twisting and $g\alpha\,\partial T/\partial y$ by horizontal temperature gradient. Diffusion and convection may also be present in the front, and Welander (1963) has shown how these determine a thickness for it, again of order $\sqrt{(\nu/f)}$. The two fundamental rates of creation must in the meantime balance on vertical integration through the layer, giving

$$f\Delta u = -(g\alpha\Delta T)\tan\epsilon \tag{6}$$

because $\int(\partial T/\partial y)\,dz$ through a front at angle ϵ to the horizontal is $(\Delta T)\tan\epsilon$.

An application of the balancing of diffusion against *stretching of vertical vorticity* is seen in the Ekman layer below a cyclone, involving the characteristic and well-known convergence. In a cyclone the vertical vorticity exceeds the planetary value f, but tends to f at the ground. There is thus a diffusion of vertical vorticity into the ground, which in a steady state must be balanced by a continual stretching of the vertical component of vorticity by an upflow w_0 above the Ekman layer:

$$\nu(\partial\zeta/\partial z)_{z=0} = fw_0. \tag{7}$$

For Ekman layers with constant ν this gives a relationship

$$\zeta_0\sqrt{(\nu/2f)} = w_0 \tag{8}$$

between the external vorticity and velocity components perpendicular to an Ekman layer, which both is a quantitative measure of the convergence and is important also in the next subject I want to turn to.

4. Fat bodies of rotating fluid

This is the study, whether theoretically or in the laboratory, of fat bodies of homogeneous fluid in rigid rotation, say about the z-axis, so that in the undisturbed state the vorticity is everywhere

$$(0, 0, 2\Omega). \tag{9}$$

The classical result in this field is the Taylor–Proudman theorem that, if viscosity can be ignored, any small *steady* disturbance flow field must in the limit of zero Rossby number be independent of z.

This is essentially because any variation in the velocity field with z would change the magnitude or direction of the arrows representing the undisturbed vorticity vector (9), and in steady flow with velocity of order of magnitude U varying significantly over distances of order L, with the Rossby number

$$U/2\Omega L \tag{10}$$

very small, no slow convection of the small relative vorticity could balance this change in the large undisturbed vorticity. The velocity field therefore takes the form

$$(u(x,y), v(x,y), w(x,y)) \tag{11}$$

and the equation of continuity is

$$\partial u/\partial x + \partial v/\partial y = 0. \tag{12}$$

The general motion is a combination of a two-dimensional motion in planes perpendicular to the z-axis with a motion purely parallel to the z-axis with velocity independent of z.

If the body of fluid is bounded above and below by surfaces $z = H(x,y)$ and $z = h(x,y)$, then boundary conditions of inviscid type at these surfaces can be satisfied with equations (11) and (12) only if

$$u = \partial\psi/\partial t, \quad v = -\partial\psi/\partial x, \quad H - h = F(\psi). \tag{13}$$

Thus, vertical rod-shaped elements remain vertical and move along paths

$$H(x,y) - h(x,y) = \text{const.}, \tag{14}$$

so that their lengths remain unaltered. This is obviously necessary to avoid changes in the undisturbed vorticity that, as we saw, could not in steady flow be balanced by effects of the disturbances convecting their own vorticity. These motions are sometimes described as purely two-dimensional, but actually the w-velocity is absent only under the rather special condition when the altitude contours

$$H(x,y) = \text{const.}, \quad h(x,y) = \text{const.} \tag{15}$$

of ceiling and floor are the same curves.

It is possible for an inviscid solution of this kind to be valid in most of the flow, excepting Ekman-type boundary layers on the floor and the ceiling, provided that the Taylor number

$$\Omega L^2/\nu \tag{16}$$

is large, because then the Ekman-layer thickness, of order $\surd(\nu/\Omega)$, is small compared with the dimension L of the system. These Ekman layers make only a slight difference to the boundary condition appropriate to the internal inviscid flow. We saw in fact that there is a small normal outflow (8) from an Ekman layer proportional to the normal component of the disturbance vorticity outside the layer. When $H-h$ varies substantially as a function of the stream function ψ, cyclonic vorticity generating outflow from both floor and ceiling boundary layers prevents streamlines from accurately following curves $H-h =$ const. and causes them to sidle towards slightly higher values of $H-h$; while anticyclonic vorticity moves a streamline towards slightly lower values.

Such 'relief' is impossible in the other extreme case, treated recently by Jacobs (1964), when $H-h$ takes the same value on all streamlines throughout an extended region of the (x, y)-plane. In such a region the inviscid relative flow must, he argues, be irrotational, since any cyclonic vorticity would produce outflow from both Ekman layers and so would continually shrink vortex lines, in a manner impossible in steady flow. The argument says essentially that, for example, between parallel planes rotating at angular velocity Ω, the disturbed flow must have an axial vorticity component neither more nor less than 2Ω. It may seem a little artificial, but only until we remember that outflow from Ekman layers is due to diffusion of normal vorticity into the boundary. Evidently, any excess vorticity would diffusively drain away, and so in the steady-flow limit would be absent. The smaller ν is, the slower this would happen, in agreement with Jacobs's statement that the limit as $\nu \to 0$ and $t \to \infty$ is a non-uniform one.

The whole situation is related to the fact observed earlier that thin-sheet flow for constant Coriolis parameter is degenerate in the limit of zero Rossby number. In this problem, similarly, *any* two-dimensional flow satisfying (12) might at first sight represent a steady solution, and only such refined considerations as the finite thickness of Ekman layers can be used to select the true one. Unfortunately, other refined considerations like finite Rossby number, or slight non-uniformity of sheet thickness, could perhaps alter this determination!

But I cannot any longer avoid mentioning the startling flow behaviour predicted in any region where $H-h$ is substantially smaller* than on the streamlines approaching that region. These streamlines cannot penetrate the region and must flow round it. The 'Taylor column' over the region can, however, contain autonomous streamlines, presumably conforming to the same rule (13), and separated from the external streamlines by a detached cylindrical shear layer. Analysis indicates that such shear layers, in which, for example, stretching and twisting of undisturbed vorticity can respectively be balanced by diffusive action on the large gradients of the horizontal and vertical components of tangential velocity, are of somewhat complex structure.

Experiments at small Rossby number (see, for example, Hide & Ibbetson 1965) have confirmed the existence of Taylor columns and the approximate two-dimensionality of the flow outside the shear layers, but have shown that irrotationality of the flow around the column is not very accurately achieved, partly

* Similar phenomena occur when 'smaller' (used here to fix the ideas) is replaced by 'larger'.

because the detached cylindrical shear layer is capable, as might be expected, of sucking in or expelling fluid.

Other motions in fat bodies of rotating fluid that have been discussed are driven, not by any external pressure gradients, but by direct viscous stress at boundaries. I shall briefly discuss these from the point of view of changes of absolute vorticity.

The results of Greenspan & Howard (1963) and Greenspan (1964) on 'spin-up' of a fluid in steady rotation, due to a sudden small increase in the angular velocity of both floor and ceiling, can be interpreted by saying that initially radial vorticity is produced at the boundaries but this generates azimuthal vorticity as described in §3 and, after a few rotations, Ekman layers are formed. The *anti*-cyclonic relative vorticity outside these layers causes *in*flow which stretches the external vorticity. This must reach a value characteristic of the new angular velocity of the boundaries in a time of order $(L^2/\nu\Omega)^{\frac{1}{2}}$, which is found to be in good agreement with experiment.

When the floor and the ceiling rotate at different speeds, any steady motion that is set up has to involve an inviscid core rotating at an intermediate speed (Batchelor 1951), determined by the condition that the outflow from the Ekman layer on the boundary with respect to which it is cyclonic is identical, for each (x, y), with the inflow into the boundary with respect to which it is anticyclonic. The still more complicated case, considered by Proudman (1956) and Stewartson (1966), of flow between concentric spheres rotating at different speeds, combines these considerations of inflow and outflow balance with 'Taylor column' considerations.

5. Wind-driven steady ocean currents

I shall now continue my survey of the role of absolute vorticity in steady motions with a discussion of vorticity generated by non-conservative external forces, in the case of steady ocean currents driven by that part of the wind-stress field which represents its long-period average.

Many of the writers on this subject have preferred to idealize the ocean as a 'thin sheet' of liquid with the external wind-stress field of force distributed through the depth of the liquid. Their results, which I shall mention first, are strongly influenced by the degeneracy of thin-sheet problems with constant Coriolis parameter in the limit of zero Rossby number, a limit to which typical situations are closer in oceanography than in meteorology. It is really because, in this limit, the whole ensemble of three-dimensional 'fat-body' solutions consists of effectively two-dimensional solutions satisfying (12), that the ensemble of two-dimensional solutions is far more extensive than in non-degenerate problems.

Another symptom of degeneracy is that the presence of a forcing term can lead to no solution at all. Thus, if the distributed force per unit mass equivalent to the applied wind stress is $(X, Y, 0)$, then from §2 the vertical component of vorticity must change at a rate

$$\partial Y/\partial x - \partial X/\partial y, \tag{17}$$

and in steady thin-sheet flow with constant Coriolis parameter at zero Rossby number there is no way of balancing this rate of change of vorticity due to wind-stress curl.

Because the problem is degenerate, every slight departure from the assumptions may need to be studied. However, the classical solution to the problem of what really happens is Sverdrup's, that the steady solution is dominated by the variation of Coriolis parameter with latitude, so that (17) is balanced by the rate of change of planetary vorticity

$$Df/Dt = vdf/dy = \beta v, \tag{18}$$

where v is the northward component of current.

Because the coefficient

$$\beta = 2{\cdot}3 \times 10^{-13} \cos\theta \,\mathrm{cm}^{-1}\,\mathrm{sec}^{-1} \tag{19}$$

is so small, it is necessary to watch carefully all other terms that might be included in the balance. The neglect of unsteady terms, for example, is justified only if the driving forces are effectively constant, not just over one day as in elementary meteorological problems, but over a period $1/\beta L$, where L is the scale of the disturbance; and this is a week even if L is as big as 1000 km. This remark in no way detracts from the value of calculating steady response to, for example, a seasonal average wind stress over a given ocean, but it does mean that unsteady response to gradual changes of wind stress is also of great importance.

I shall illustrate the problem of applying boundary conditions in the steady case by considering a seasonal average wind-stress of the simple zonal-wind form $(X(y), 0, 0)$, giving

$$v = -\beta^{-1}X'(y), \quad \partial u/\partial x = \beta^{-1}X''(y). \tag{20}$$

Obviously if $X''(y) > 0$ there is inflow into at least one of any two meridional ocean boundaries, and conditions of zero normal flow cannot be satisfied on both. This difficulty is resolved by postulating a boundary layer receiving all this inflow of slowly moving fluid and converting it into a relatively fast flowing boundary current.

In this current changes in vertical vorticity due to north–south movement of planetary vorticity were at first supposed to be balanced by diffusion from the boundary. The idea that horizontal diffusion coefficients could produce something as wide as the Gulf Stream depended, of course, on the view that those currents generated by random wind-stress fluctuations with periods up to a week or more would contribute (non-linearly) to diffusion, as well as ordinary turbulence. Many considerations combined, however, to show that convection of *relative* vorticity cannot be ignored in such boundary currents and may dominate over diffusion in many parts of them. Such parts are called 'inertial boundary layers' (Fofonoff 1954; Morgan 1956; Carrier & Robinson 1962; Veronis 1963).

A layer with inflow has fluid entering slowly and with negligible relative vorticity. Poleward flow then gives it anticyclonic vorticity, while at the same time it gets well inside the layer and can speed up. This is geometrically in agreement with the distribution of shear for a western boundary only (figure 2). Similarly, equatorial flow gives it cyclonic vorticity which again fits the necessary shear distribution on a western but not on an eastern boundary. We conclude that a boundary current with inflow can occur only on a western boundary. The equation (20) should be solved, therefore, so that the eastern boundary is a streamline, and the inflow is into the western boundary.

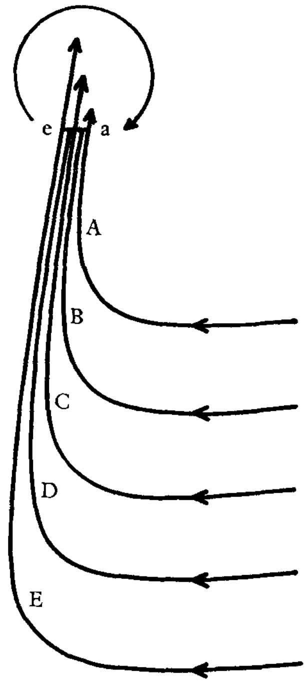

FIGURE 2. Inertial boundary current with inflow. A, B, C, D, E: Slow flow enters poleward-moving inertial layer on western boundary and begins to be accelerated. ae: Velocity profile of resulting sheared flow possesses anticyclonic vorticity.

The same arguments have been used in reverse to suggest that when outflow is expected on at least one boundary, as is so in the simple case here discussed when $X''(y) < 0$, the necessary boundary layer with outflow can be accommodated only on the eastern boundary,* and therefore that the slow Sverdrup flow outside the layers is invariably towards the west, to yield inflow into the western boundary current and outflow from the eastern one. But this conclusion raises two questions: 'how the fluid gets back east again'; and perhaps also 'how at a latitude where $X''(y)$ changes sign the transition between a western current with inflow and an eastern current with outflow occurs'.

Carrier & Robinson (1962) gave a tentative answer to both questions using the idea of a eastward-moving free inertial jet, carrying away in a concentrated form the fluid that had flowed into the western boundary current and delivering it at the eastern boundary. To an ordinary laboratory fluid dynamicist it may seem unthinkable that a jet of some 100 km wide might retain its identity over 5000 km of ocean, because of diffusion of vorticity and entrainment of new fluid,

* But two difficulties in the maintenance of eastern boundary currents are noted at the end of this section.

but on this model it can happen because any new fluid entrained from the pole-ward side acquires from its displacement cyclonic relative vorticity which enhances the jet shear (and similarly anticyclonic from the equatorward side) and this seems to make it possible for actual intensification of the jet to occur, while the growth in thickness is very slow (on a simple laminar calculation, ultimately like $x^{\frac{1}{2}}$, by contrast with $x^{\frac{2}{3}}$ in the non-rotating case*). Neither the boundary current nor the inertial jet is significantly influenced by the local wind stress. These ideas, which may still not be in a fully satisfactory form, throw light on early experiments of Fultz & Long (1951) in rotating spherical shells of liquid, which showed that objects moving eastward relative to the fluid drew out behind them concentrated jet motions, although objects moving westward did not.

Now I have so far given no attention to the fact that the true wind stress, far from being uniformly distributed over the depth of the ocean, is concentrated at the surface. The difficulties arising from this, as well as from the stratification of the ocean and the topography of its bottom, are substantial. In discussing them it is necessary to remember the essential degeneracy of the problem, which may mean that numerous small effects need to be studied carefully to see whether they might alter the conclusions.

The wind-stress (τ_x, τ_y) feeds in horizontal vorticity at the surface, and, more important for us, feeds in also vertical vorticity at a volume rate per unit area per unit time

$$\rho^{-1}(\partial\tau_y/\partial x - \partial\tau_x/\partial y). \tag{21}$$

This, then, is the rate of change of the total vorticity integrated through the depth of the ocean, and, if it is to be balanced by the beta-effect, it must be equal to βV, where V is the integral of the northward velocity v with respect to depth.

Diffusion carries vorticity downward only through an Ekman-layer thickness of order $\surd(\nu/f)$, but we saw in §4 that inflow into the Ekman layer can alter the 'internal' flow beyond it. If the Coriolis parameter were constant, the feeding in of cyclonic vorticity at the surface would have to be balanced by upward inflow w_0 such that fw_0 equals (21), exactly as in rotating-disc problems or cyclonic convergence. A vertical current four orders of magnitude smaller than the postulated northward flow would suffice. It would, however, have to continue until it had increased the vorticity in the depths below the Ekman layer to such a high cyclonic value that the *bottom* Ekman layer would provide a balancing outflow. This task would appear a Herculean one because near the bottom eddy viscosity is probably quite small, so that extremely large cyclonic vorticity would be needed to produce the required outflow.

* A solution

$$\psi = (\beta\nu^3x^3)^{\frac{1}{4}} F[y(\beta/\nu x)^{\frac{1}{2}}]$$

exists for the stream function on boundary-layer assumptions, where $F(\eta)$ is a function tending exponentially to zero as $\eta \to \pm\infty$ and satisfying

$$F^{\text{iv}} + \tfrac{3}{4}FF''' - \tfrac{1}{4}F'F'' + \tfrac{1}{2}\eta F' - \tfrac{3}{4}F = 0.$$

Other solutions, valid when x, the co-ordinate measured in the direction of the current, is not so large, are given by Long (1960).

The mechanism in question is, however, completely swamped by the 'beta-effect'. With variable Coriolis parameter, there is no need for inflow into the surface Ekman layer to be balanced by outflow from a bottom layer; in fact, the vertical velocity w is not in general independent of z between the two Ekman layers, because a rate of generation of cyclonic vorticity $f\partial w/\partial z$ by stretching can be balanced by βv. If this result can be integrated from the surface down to a bottom where w is effectively zero, we regain the relation which equates (21) to βV. This integrated relationship is of great value, even though it leaves us still tantalizingly short of information on the vertical distribution of v, which must depend critically on stratification, being influenced, for example, by equation (4).

The influence of the bottom is, on the other hand, certainly not unimportant, and inertial currents at any rate are not places where w can be assumed zero there. A western boundary current is helped by bottom topography (Warren 1963; Greenspan 1963) to cling to depth contours since, for example, a poleward current on moving into shallower water must contract the planetary vorticity and therefore cause anticyclonic curvature of the stream, taking it out of the shallower water; and vice versa.

On the other hand, an eastern boundary current would show exactly the opposite effect, which may be one of the explanations why regular boundary currents do not form up well on eastern boundaries, especially those of irregular shape. Another explanation is that diffusion, also, is more of a hindrance to maintaining concentrated flow attached to the shoreline in eastern than in western boundary currents.

6. Wave motions for constant Coriolis parameter

But it would be misleadingly unbalanced to continue any longer on a survey of the dynamics of rotating fluids without discussing waves and wavy movements. The wave systems that are possible as small disturbances to a steady state of a fluid are all modified by rotation, which, in addition, makes more than one new type of wave system possible. Most unsteady motions of rotating fluid are strongly influenced by the properties of these wave systems, for example, by their generation, dispersion, attenuation and non-linear interaction; and even the characteristics of many steady flows are determined in important respects by them.

It is necessary *both* to understand the elementary disturbances of (roughly speaking) plane-wave type that are possible, *and* to be able to predict how they combine in more complicated motions. In the time that remains to me I shall be able to survey rapidly the main elementary wave-like solutions that are known, making the account as connected as possible by vorticity considerations as before, and then to explain and illustrate just one main technique for predicting how they combine; actually, the technique for calculating the wave pattern generated by a moving disturbance. I chose *this* technique to describe and illustrate, out of several that might have been mentioned, because the problem is important in so many contexts, for example, in studying ocean movements set up by a travelling atmospheric disturbance, *or* atmospheric wave motions generated by the passage of wind over a large topographical feature*, *or*, in fat bodies, disturbances moving either normal to the axis or along it, with or without formation of Taylor columns.

It is with the elementary waves possible in 'fat bodies' of homogeneous incompressible fluid in rotation that my general survey of waves begins. These waves are usually called inertial waves, which is probably an appropriate enough name for waves propagated following the basic rule governing vorticity changes, because that rule is derived from the fundamental laws about rotary inertia.

The propagation can be understood either analytically or geometrically, according to taste. Analytically, small changes in vorticity must satisfy

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{v} = 2\Omega\frac{\partial \mathbf{v}}{\partial z}, \tag{22}$$

representing stretching of the undisturbed vorticity vector (9). Applying the operation on the left a second time, we obtain

$$-\frac{\partial^2}{\partial t^2}\nabla^2\mathbf{v} = 4\Omega^2\frac{\partial^2\mathbf{v}}{\partial z^2}, \tag{23}$$

which has plane-wave solutions

$$\mathbf{v} = \mathbf{v}_0\exp[i(-\sigma t + lx + my + nz)] \tag{24}$$

if

$$\sigma^2(l^2+m^2+n^2) = 4\Omega^2 n^2. \tag{25}$$

Thus there is a high-frequency cut-off at the semi-diurnal frequency 2Ω. The group velocity

$$(\partial\sigma/\partial l, \partial\sigma/\partial m, \partial\sigma/\partial n) \tag{26}$$

is at right angles to the wave-number vector (l, m, n), and so is parallel to wave crests instead of perpendicular to them.

Geometrically, we begin by noticing that a plane wave in an incompressible fluid can involve only motions in the planes of propagation. But there will in general be a time lag, say τ, between adjacent planes and this causes (figure 3) an arrow-shaped element of fluid of length ϵ in the direction of the undisturbed vorticity to change by $\mathbf{V}\tau$, in the direction of the fluid velocity $\mathbf{V}$, giving additional vorticity

$$2\Omega(\mathbf{V}\tau/\epsilon)$$

in the plane. The associated shear must be in the velocity component at right angles, but this velocity gradient is $\mathbf{A}\tau/\epsilon\sin\theta$, where $\mathbf{A}$ is the acceleration of a fluid particle and $\epsilon\sin\theta$ is the distance between the planes. Hence the acceleration $\mathbf{A}$ is at right angles to the velocity $\mathbf{V}$ and of proportional magnitude $2\Omega V\sin\theta$, and so particle paths are circles, and the radian frequency is $\sigma = A/V = 2\Omega\sin\theta$, in agreement with (25).

* In this written version of the lecture, only these two cases have, briefly, been treated. The quite general treatment included in the spoken version, together with many applications, including those to Taylor column formation, is being separately published in an expanded form (Lighthill 1966).

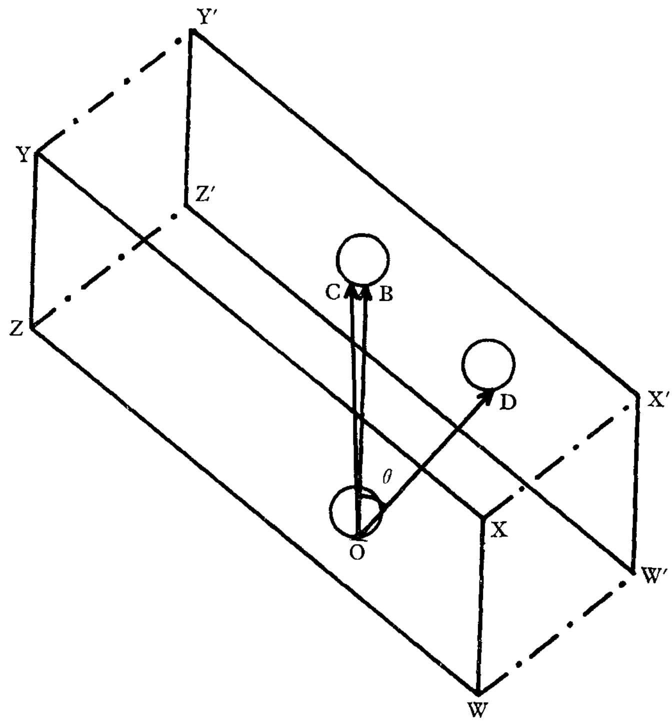

FIGURE 3. Illustrating plane-wave propagation of inertial waves. WXYZ and W′X′Y′Z′: planes of constant phase; OB: Undisturbed position of arrow-shaped element of length ϵ, representing undisturbed vorticity vector $(0, 0, 2\Omega)$; OC: disturbed position of element, when particles of fluid on W′X′Y′Z′ are describing same curves as those on WXYZ but after time lag τ; BC: this element of length $\mathbf{V}\tau$, where $\mathbf{V}$ is velocity, therefore represents relative vorticity of disturbance; OD: perpendicular between planes, of length $\epsilon \sin\theta$, where θ is angle BOD between OD and z-direction. Since difference of velocity at O and D is $\mathbf{A}\tau$, where $\mathbf{A}$ is acceleration of fluid element, shear is $\mathbf{A}\tau/\text{OD} = \mathbf{A}\tau/\epsilon \sin\theta$. This must have magnitude equal to the vorticity and direction perpendicular to it.

We can then interpret the fact that the group velocity vector lies in the plane by saying that the energy propagation process is one in which the different circular paths of elements in the plane get pulled into phase, along the direction specified by resolving the axis of rotation on to the plane. Experiments by Oser (1957) demonstrated that energy is propagated at an angle $\frac{1}{2}\pi - \theta = \cos^{-1}(\sigma/2\Omega)$ to the axis of rotation by showing that the waves from a local source of frequency σ fill a cone with this semi-angle. The group velocity becomes zero only when the plane is at right angles to the axis of rotation, that is, when $\theta = \frac{1}{2}\pi$ and $\sigma = 2\Omega$. Standing oscillations are possible at this frequency. In the other limit $\theta \to 0$ (that is, $\sigma/2\Omega$ small) the *phase* velocity tends to zero, but there is steady propagation of wave energy along the axis of rotation at a group velocity $2\Omega/\sqrt{(l^2 + m^2)}$, and this can be thought of as the speed of formation of the Taylor column, in rather good agreement with the results of experiments by Brooke Benjamin & Barnard (1964).

Now consider homogeneous fluid with a free surface $z = 0$ and, maybe, also a bottom $z = -H$. Equation (25) still specifies the waves that will occur for

$\sigma < 2\Omega$ with l, m and n real. The free surface, however, makes possible, in addition, solutions with n imaginary:

$$\sigma > 2\Omega, \quad n = iN, \quad N^2(\sigma^2 - 4\Omega^2) = \sigma^2(l^2 + m^2). \tag{27}$$

These solutions, varying exponentially with z, can satisfy the boundary conditions if combined into a solution proportional to $\cosh N(z+H)$, where

$$\sigma^2 = gN \tanh NH. \tag{28}$$

Equations (27) and (28) show that for $\sigma > 2\Omega$, when inertial waves propagating internally are impossible, surface waves have become possible, with their dispersion relation modified by rotation. Such surface waves have a *low*-frequency cut-off at the semi-diurnal frequency.*

When the wavelength is much greater than the depth H, we can make the tidal approximation NH small, so that fluid movements are approximately horizontal, and constant from the surface to the bottom. The equations then give

$$\sigma^2 = 4\Omega^2 + gH(l^2 + m^2). \tag{29}$$

In this discussion the axis of rotation was taken vertical. Once the motions are approximately horizontal, however, the theory can be applied to tides at any latitude on a rotating earth, because (as in the earlier meteorological discussions) the vertical component of vorticity varies independently of the horizontal component. Equation (29) becomes

$$\sigma^2 = f^2 + gH(l^2 + m^2), \tag{30}$$

because the undisturbed value of vertical vorticity is f.

The *disturbed* value of vertical vorticity is f times the ratio of depth to undisturbed depth. The presence of such vorticity makes possible solutions in which yet another component of wave-number is a pure imaginary quantity, say $m = iM$. For one can satisfy the inviscid boundary conditions at a sea coast by means of a vortex sheet, with vorticity falling off exponentially like $\exp(-My)$ with distance y from the coast. The vorticity in this sheet propagates one-dimensionally along the coast by rises and falls in depth, and this is the mode of tidal propagation known as the 'Kelvin wave'.

The other important waves influenced by gravity are the internal waves, which like the inertial waves are propagated through 'fat bodies' of fluid.† Their analysis in the presence of rotation can start from equation (22), with an additional rate of production of horizontal vorticity (2) due to temperature gradient included on the right-hand side. The z-component of equation (23) then becomes

$$-\frac{\partial^2}{\partial t^2}\nabla^2 w = 4\Omega^2 \frac{\partial^2 w}{\partial z^2} - g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial T}{\partial t}$$
$$= 4\Omega^2 \frac{\partial^2 w}{\partial z^2} + N^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \tag{31}$$

* In applying these results to laboratory experiments, the possible errors resulting from depth non-uniformity due to centrifugal force need, however, to be carefully considered (Miles 1964*a*).

† Only a body whose vertical dimension is small compared with the scale height is considered here. For the more general case, see Tolstoy (1963).

where N is the Väisälä–Brunt frequency, defined by

$$N^2 = g\alpha(d\Phi/dz), \tag{32}$$

and Φ is the undisturbed 'potential temperature', which satisfies

$$\partial T/\partial t = -w\, d\Phi/dz \tag{33}$$

for small disturbances.

Plane-wave solutions of (31) satisfy

$$\sigma^2(l^2+m^2+n^2) = 4\Omega^2 n^2 + N^2(l^2+m^2), \tag{34}$$

and therefore exist only when the frequency σ *lies between* 2Ω and N. If stratification were so weak that $N < 2\Omega$ this would simply add a low-frequency cut-off at $\sigma = N$ to the existing high-frequency cut-off for inertial waves at $\sigma = 2\Omega$.

However, in the cases of geophysical interest, when $2\Omega \ll N$, the character of inertial waves is completely destroyed by stratification. No waves for $\sigma < 2\Omega$ now exist at all, and the waves for $\sigma > 2\Omega$ are just modified gravity waves in which, as for the tides, rotation produces a *low-frequency* cut-off at $\sigma = 2\Omega$. I have made these arguments for the case when the axis of rotation is vertical, but for any direction of the axis of rotation study of equation (34) with $4\Omega^2 n^2$ replaced by $[2\mathbf{\Omega}.(l, m, n)]^2$ shows again that inertial waves with σ substantially less than 2Ω are completely destroyed by such stratification as is undoubtedly present in geophysical cases.

Stern (1963) has argued that a rotating spherical shell of liquid could sustain wave modes trapped near the equator, and Bretherton (1964) explained these modes as due to continued reflexion between the boundaries of inertial waves of low frequency whose group velocity makes a small angle with the axis of rotation. But in a geophysical problem the argument suggests that stratification would prevent these waves from occurring.

Equation (34) with N^2 negative is relevant to problems of unstably stratified fluids. The term due to rotation cannot make the system stable, but it does cause the Rayleigh number, below which dissipative effects can cancel the gravitational instability, to increase (Chandrasekhar 1953; Chandrasekhar & Elbert 1955; Fultz 1959).

7. Wave motions for variable Coriolis parameter

I must turn attention now to waves in systems with variable Coriolis parameter, beginning with motions that are purely horizontal with zero divergence. Such divergenceless motions can be expected to be realized at any rate in experiments with spherical shells of liquid, like those of Fultz & Long (1951) and Frenzen (1955). Although I mentioned earlier that eastward-moving objects in their experiments drew out behind them concentrated jet motions, westward-moving objects by contrast set up wavy disturbances. These are the waves associated with the names of Rossby and of Haurwitz.

Their law of propagation states that the vertical component of relative vorticity changes simply due to north–south convection of planetary vorticity:

$$\partial\zeta/\partial t = -\beta v. \tag{35}$$

On Rossby's 'beta-plane', locally approximating the true spherical-surface metric by a flat one, this gives for the stream function

$$\partial(\nabla^2\psi)/\partial t + \beta(\partial\psi/\partial x) = 0. \tag{36}$$

Plane-wave solutions like (24) are then possible if

$$\sigma(l^2 + m^2) + \beta l = 0. \tag{37}$$

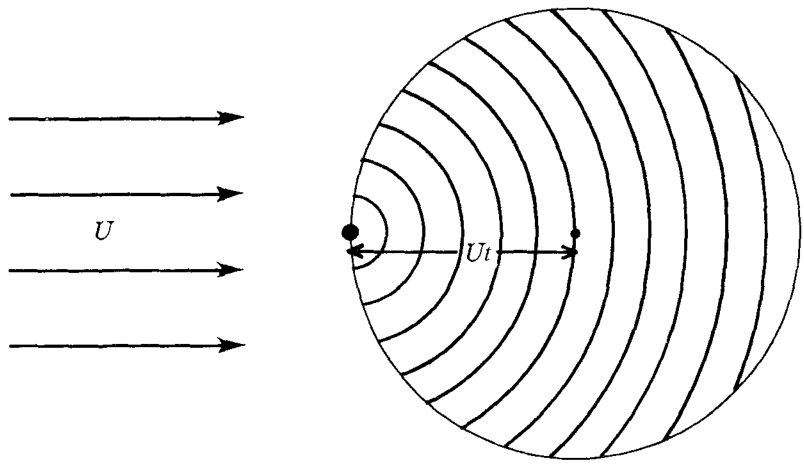

FIGURE 4. Rossby waves generated by an obstacle in an eastward flow with uniform velocity U that began at time $t = 0$. The waves are of uniform length $2\pi(U/\beta)^{\frac{1}{2}}$ and spread with uniform group velocity U relative to a point moving with the flow. After time t, therefore, they fill a circle (shown faint) of radius Ut, centred on a point Ut downstream of the obstacle.

The speed of drift of the whole wave *pattern* towards the west is therefore

$$-\sigma/l = \beta/(l^2 + m^2) = \beta\lambda^2/4\pi^2, \tag{38}$$

and depends only on the wavelength λ. The group velocity $(\partial\sigma/\partial l, \partial\sigma/\partial m)$ has the same magnitude (38) but, as Longuet-Higgins (1964) has pointed out, is in a different direction and makes an angle with the eastward direction which is twice the angle that the phase velocity makes (or the wave-number vector (l, m)).

The drift can, of course, be cancelled by a westerly flow U equal to (38), which, for example, is 4 m/sec for a wavelength of 3000 km at 45° latitude. On such a flow, therefore, stationary waves can exist, satisfying

$$\nabla^2\psi + (\beta/U)\,\psi = 0. \tag{39}$$

Any source, such as an obstacle in the flow, can according to this equation generate waves spreading out from it isotropically, with uniform wavelength. Also the group velocity of these waves, relative to the flow, is U, so that, for example, if the flow had started at time $t = 0$ they would be present only in a circle centred on a point a distance Ut downstream of the obstacle, with radius Ut (figure 4).

Conversely, a source moving towards the west through still air with speed U would also generate such waves. By contrast, the oceanographers are interested in the case when an atmospheric disturbance passes eastward over the ocean, which to this approximation is represented by equation (39) with U negative, with non-wavy solutions* decaying exponentially beyond a distance $\surd(-U/\beta)$. Rossby waves in the ocean cannot be excited by such eastward-moving disturbances.

In the true spherical geometry, which must be used for the really large-scale phenomena, equation (35) can be written

$$\frac{\partial}{\partial t}\nabla^2\psi + 2\Omega\frac{\partial\psi}{\partial\phi} = 0, \tag{40}$$

where in terms of latitude θ and longitude ϕ the Laplacian signifies

$$\nabla^2\psi = \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\cos^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{41}$$

and the velocities are

$$u = \frac{1}{R}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{1}{R\cos\theta}\frac{\partial\psi}{\partial\phi}. \tag{42}$$

Evidently, waves can again be generated by disturbances moving westward. But, because the sphere is a limited region, there are certain velocities at which the disturbance can move for which these waves get intensified by resonance. A normal mode on the sphere,

$$\psi = S_n\left(\theta, \phi + \frac{2\Omega t}{n(n+1)}\right), \tag{43}$$

where S_n is a spherical harmonic satisfying

$$\nabla^2 S_n + n(n+1)\,S_n = 0, \tag{44}$$

is such a solution, which a disturbance moving westward at an angular speed

$$\Delta\Omega = \frac{2\Omega}{n(n+1)} \tag{45}$$

about the axis would make. Long (1952) found that well-marked patterns were created in the spherical-shell experiments for speeds of westward movement satisfying just such a condition, but with Ω on the right replaced by $\Omega + \Delta\Omega$, apparently as if in the steady state almost the full angular velocity $\Omega + \Delta\Omega$ of the obstacle had communicated itself to the fluid.

Longuet-Higgins (1964) points out that some of the resonant solutions involve wavy motion only at the lower latitudes; for example, the spherical harmonic P_n^s for large n does so only below the latitude $\cos^{-1}(s/n)$, where the wave crests reflected back at a locus of cusps. Modes of this kind were observed by Fultz & Frenzen (1955).

A more recent paper by Longuet-Higgins (1965) shows (amongst other things) that a divergence-free approximation neglecting tidal terms is in geophysical problems truly applicable only under very restrictive conditions. This result is yet another consequence of the degeneracy of the steady divergence-free case,

* Typically, $K_0[r\surd(-\beta/U)]$, where r is distance from the source and K_0 is a modified Bessel function of the second kind.

which means that many terms which would normally be expected to be small are in fact important. The 'beta-effect' is only one of these. Equation (37) shows that a very slight unsteadiness, with a frequency of order βL, where L is a characteristic length, is equally important (as I mentioned already in relation to wind-driven currents). But also the tidal contribution, which might be expected to be small if velocities of drift are small compared with $\sqrt{(gh)}$, is *not* in fact small and needs to be added to (37), giving

$$\sigma[l^2+m^2+(f^2-\sigma^2)/gH]+\beta l = 0. \tag{46}$$

If σ is of order βL we may, actually, neglect the additional term in σ^2, but not that in f^2. Then the speed of drift of the Rossby wave pattern towards the west becomes

$$U = \frac{\beta}{l^2+m^2+f^2/gH}; \quad \text{giving} \quad l^2+m^2 = \beta/U - f^2/gH. \tag{47}$$

The two terms here would be actually *equal* at a latitude of 45° only if U was as great as 16 m/sec multiplied by the depth in kilometres. In practical cases, therefore, the tidal term is a minor, but usually not negligible, correction to the wavelength. The whole theory of Rossby waves, including the case shown in figure 4, remains true if this correction is made.

With spherical geometry the corresponding equation is

$$\frac{\partial}{\partial t}\left[\nabla^2 V - \left(\frac{4\Omega^2R^2}{gH}\sin^2\theta\right)V\right]+2\Omega\frac{\partial V}{\partial \phi} = 0, \tag{48}$$

where $V = v\cos\theta$ is what would be $R^{-1}\,\partial\psi/\partial\phi$ for a divergenceless case. Here $4\Omega^2R^2/gH$ is about 90 divided by the depth in kilometres, so that the additional term in (48) is comparable with $n(n+1)\,V$ for the smaller values of n, and the pattern drifts to the west slower for these than (45) would predict, just as (47) gave in the beta-plane case.

But even a very rapid survey of waves, like this, must include discussion of waves in sheared flows. In these, the vorticity of the undisturbed flow shows a variability additional to that exhibited by the Coriolis parameter. One problem studied by the meteorologists has been that of disturbances to a zonal flow whose speed varies only with latitude. The problem is of limited applicability, because it is founded on the barotropic assumption of uniform potential temperature, excluding variation of wind with height; but it serves as a useful introduction to the principles.

If the vertical component of relative vorticity in the undisturbed state is $Z(y)$, then equation (35) is augmented by the additional term $-Z'(y)\,v$. When we work in a beta-plane this term is $+U''(y)\,v$, giving

$$\left[\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right]\nabla^2\psi+[\beta-U''(y)]\frac{\partial\psi}{\partial x} = 0. \tag{49}$$

Then waves whose x-component of wave-number is α have $\psi = \phi(y)\exp[i\alpha(x-ct]$, where

$$(U-c)(\phi''-\alpha^2\phi)+(\beta-U'')\phi = 0. \tag{50}$$

The beta-effect helps to stabilize a shear flow that would normally be unstable; and for $U = U_0 \tanh(y/L)$ Lipps (1965) found the condition for stability to be

$$\frac{U_0}{\beta L^2} < \frac{4}{3\sqrt{3}}. \tag{51}$$

The baroclinic instability problem has proved far more difficult, although its significance is so much greater. At this stage in a lecture I can perhaps be forgiven for omitting the mathematical details (see, for example, Charney 1947; Kuo 1952; Arnason 1963; Pedlosky 1963, 1964; Miles 1964*b*; Phillips 1964). The waves considered are dependent both on the beta-effect and on vertical wind shear, with its associated horizontal temperature gradients, but do not allow for any other variation with latitude in the undisturbed quantities or their perturbations. In certain ranges of wave-number they can be thought of as Rossby waves that derive a certain rate of exponential build-up from the destabilizing effect of the wind shear, mitigated to some extent by the stabilizing effect of temperature stratification.

But I know I have talked for far too long, and now it is time to yield the floor to experts.

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